Finding differential equations that describe the motion of a mechanical system

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1 Introduction

The intended audience of this article are non-physicists with some knowledge of calculus and simple (particle) mechanics on the level of an average high-school physics course.

The article's scope is to introduce means of describing mechanical systems by its energy terms and deducing the differential equations that describe the system's motion.

The matter is presented in a cookbook-like fashion omitting any mathematical proofs. For a more coherent introduction to analytical mechanics, the interested reader may refer to *L.D. Landau, E.M. Lifschitz:* "Course of Theoretical Physics: Volume 1, Mechanics".

2 Concepts of Particle Mechanics

2.1 What is a Particle?

A particle is a piece of matter that is assumed to have infinitely small size. In many cases, bodies that have a finite size can be described as particles to a reasonable accuracy. An example of such is the motion of planets around the sun. Compared to the distances between the bodies, their size is negligible.

Therefore, particle mechanics are a good approximation for many problems that may arise, eventhough - strictly speaking - there are no particles one could observe ordinarily.

2.2 What is Energy?

Energy is something clever physicists invented to describe a certain aspect of physics. I use the verb "invented" because energy cannot be measured directly. Though everybody has a notion of what it is, one can only measure energy by indirect means.

A gravitational potential (which is a form of energy) can be measured as the force one massive body (like the earth) excerts on another (like yourself). Electrical energy heats your toaster (thus converting itself to another form of energy: heat). Chemically stored energy accelerates your car.

We will use the energy of a system of particles to describe its motion. In order to do that we need to deal with two types of energy.

2.3 Kinetic Energy

Kinetic energy is inherently connected to motion. A moving particle of a given mass has a kinetic Energy T that is proportional to the mass m and the square of the velocity v of the particle.

$$T = \frac{1}{2} \cdot m \cdot (\vec{v})^2$$

If you consider a system of n particles that do not interact with another, it becomes obvious that the kinetic energy of the whole system is equal to the sum of individual particle kinetic energies.

$$T_{total} = T_1 + T_2 + \cdots + T_n$$

This concept stems from a very fundamental idea called the "superposition principle".

The position of a particle can be described as the vector $\vec{r}$. In ordinary cartesian coordinates x, y and z, it can be written as a vector of three components:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Just like the position of a particle, its velocity $\vec{v}$ is a vector. In cartesian coordinates, that vector is equal to the total time derivatives of the components:

$$\vec{v} = \begin{pmatrix} \frac{d}{dt}x \\ \frac{d}{dt}y \\ \frac{d}{dt}z \end{pmatrix}$$

From now on, we will indicate derivatives in respect to the time t as a dot above the function to be differentiated. Thus we can write:

$$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

The formula for the kinetic energy of a particle includes the square of the particle's velocity. This is a vector product that can be calculated as follows:

$$\vec{v} \cdot \vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x} \cdot \dot{x} + \dot{y} \cdot \dot{y} + \dot{z} \cdot \dot{z} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

This is the square of the magnitide of the vector $\vec{v}$. The magnitude of $\vec{v}$ is therefore $|\vec{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

Using this, we can write the kinetic energy of a system of n particles as

$$T_{total} = \frac{1}{2} \cdot \left(m_1 \left|\vec{v}_1\right|^2 + m_2 \left|\vec{v}_2\right|^2 + \cdots + m_n \left|\vec{v}_n\right|^2 \right)$$

2.4 Potential Energy

The potential energy of a particle is connected to its position relative to the rest of the system. For example, one can increase a particle's (gravitational) potential energy by moving it away from a massive body like the earth.

The potential energy of a system of particles is usually denoted by the symbol U and characterized by an arbitrary function of the particles' coordinates. The potential energy of particle i is thus:

$$U_i(x_i, y_i, z_i) = f(x_i, y_i, z_i)$$

Using the superposition principle once again, we can write the potential energy of the system as

$$U = U_1 + U_2 + \cdots + U_n$$

In practice, many such functions only depend on the distance from some point $\vec{r_0}$. Assuming $\vec{r_0}$ to be the origin ($\vec{r_0} = (0,0,0)^T$) for the sake of simplicity, the potential energy of a particle may be written in a simpler form:

$$U(|\vec{r_i}|) = f(|\vec{r_i}|)$$

One case of potentials of the above form (which are called central potentials) is the gravitational potential:

$$U(|\vec{r}|) = \gamma \frac{m_1 \cdot m_2}{|\vec{r}|^2}$$

3 The Lagrangian Function

Getting closer to the actual differential equations of motion, we define the Lagrangian L to be the difference between the kinetic energy T and the potential energy U of the system.

$$L = T - U$$

Once the Lagrangian of a system is known, it requires just basic calculus and algebra to arrive at a differential equation. We will use the following:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

A proof of this formula can be read in *Landau* et al. If it looks a little overwhelming, consider this short explanation of the terminology: q is any coordinate (x, y, z in cartesian space), $\frac{d}{dt}$ is the derivative in respect to the time t, $\frac{\partial}{dq}$ is the partial derivative in respect to the coordinate q, and $\frac{\partial}{d\dot{q}}$ is the partial derivative in respect to the velocity $\dot{q}$ which may be any of the velocity components ($\dot{x}$, $\dot{y}$, or $\dot{z}$, again in cartesian space only).

If the problem you're trying to solve involves symmetries, it may be a better idea to use non-cartesian coordinates like spherical or cylinder coordinates. Applying the above equation to a Lagrangian yields a differential equation for the coordinates and their second (and possibly first) derivative in time. The resulting differential equations often cannot be solved algebraically. In such cases (which may be as simple as a pendulum), numeric solutions are reasonably simple to acquire using computer programs.